Two Equation Closure Model for Plasma Turbulence in a Hall Effect Thruster

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A two-equation closure model for plasma turbulence in a Hall effect thruster is derived. It is assumed that acoustic waves driven unstable by the electron Hall drift are the dominant modes propagating in the plasma and that these waves are in a turbulent, broadband state. Subject to these assumptions, a moment analysis is performed of the plasma wave kinetic equation to yield the two equation model: one relation for the evolution of the average energy density of the plasma turbulence and one for the evolution of the average rate at which this turbulence extracts energy from the electron drift. These equations are similar in form to the k-ϵ model for classical fluid turbulence. The two governing relations are related through quasilinear theory to an anomalous electron collision frequency. The resulting partial differential expressions are evaluated numerically in the state-state approximation for plasma parameter measurements taken from the channel centerline of a 9-kW class Hall effect thruster. By calibrating three fit parameters, it is found that the model predictions can be made to agree both qualitatively and quantitatively with the measured anomalous collision frequency and the measured properties of the plasma turbulence. The limitations and extensibility of the two-equation approach are discussed in the context of performing predictive, reduced fidelity simulations of Hall effect thrusters.

I. Introduction

As Hall thrusters are extended beyond their traditional operating window to both higher (> 50 kW) and lower ($< 500W$) with lifetimes an order of magnitude higher than state of the art, numerical tools will have an increasingly important role in guiding new designs and supplementing lifetime qualification efforts. To date, however, the problem of anomalous electron transport in Hall effect thrusters has been the major hurdle to developing these types of predictive numerical tools.[1] Indeed, although most plasma processes in these crossed-field devices can be explained or modeled classically, the electron dynamics are non-classical. Specifically, the amount of electron current in these devices that is able to cross magnetic surfaces is orders of magnitude higher than what would be allowed by interspecies collisions or interactions with the walls.

The non-classical nature of the electron transport recommends the use of higher fidelity model (one that goes to the atomic instead of the average fluid scale) to simulate these devices. In an analog to direct numerical simulations (DNS) that are employed in the computational fluid dynamics (CFD) community, particle in cell models or kinetic approaches that solve the species Boltzmann equation can be applied in principle to represent self-consistently the mechanisms that drive that this non-classical electron transport. The major limitation with this DNS approach is that it is computationally prohibitive. Indeed, it has only been possible to build kinetic models of full-size Hall thrusters by reducing the fidelity of the simulation by eliminating a dimension of the simulation—thus only simulating the $r-\theta$ or $\theta-z$ planes. While these reduced kinetic simulations have resulted in new physical insight into the mechanisms that may explain transport, their utility as engineering tools to predict performance and lifetime of actual Hall thruster operation is thus limited.

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In light of these challenges with DNS approaches, fluid-based or hybrid plasma models are a preferred engineering tool for the practical modeling of Hall thrusters. With shorter convergence times and minimal numerical noise, they provide the capability for rapid simulation and evaluation of different device configurations and operating conditions. The use of fluid models comes at the expense of fidelity. It has been found that due to the non-classical electron transport, classical fluid formulations such as the Branginskii closure model, cannot be applied to describe the electron dynamics in Hall thrusters.[2,3] In order to be able to increase the effective cross-field current in these simulations to be more in line with measurements, it is necessary to add additional source terms in the governing equations for the electron dynamics. The introduction of additional terms opens the governing equations (more unknowns than equations). In order to close the set of equations, a closure model, i.e. an additional set of governing equations, must be found for the new source terms.

There have been a number of attempts at identifying a closure model for the anomalous transport in fluid-based Hall thruster codes.[3–17] These begin with the assumption that the anomalous electron transport can be represented with an effect force density characterized by a scalar transport coefficient that represents either a higher effective collision frequency, or equivalently, a higher electron mobility. Different models are then proposed for this anomalous collision frequency as it depends either on position or local fluid properties. Most transport closures to date for Hall effect thrusters are zero-equations. They are algebraic expressions that depend on the background plasma properties (e.g. density, temperature, or gradients in these properties) or location in the thruster. We have summarized the most popular expressions of these in our work in Ref. 16. As an example, the simplest, and until the early 2000s, the mostly widely used expression was to assume Bohm scaling for the anomalous collision frequency.[2] While algebraic closures such as these benefit from simplicity, they have limitations either related to the quality of their predictions or their predictive capability. For instance, the approach adopted by Mikellides et al is to specify an expression for the collision frequency that depends only on spatial location.[3] While with appropriate calibration against experimental data, this model can be used to yield nearly exact matches to experimental results, this closure is not extensible to other operating conditions. On the other hand, while other algebraic closures are first-principles in that they are derived based on a direct consideration of the plasma processes driving the transport, it was shown in Ref. 16 that most of these first-principles zero equation models have limited capability to predict the actual distribution of electron transport. In an effort to overcome the limitations of these models as well as the fully empirical approach, in our most recent work, we have used machine-based discovery to identify potential algebraic closure models. The results ultimately yielded an order of magnitude improvement over several algebraic closure models employed to date, though the results admittedly were not rooted in a first-principles derivation. Indeed, more broadly speaking, it is questionable if any zero-equation models will be able predict with a high degree of fidelity the nuanced transport processes in the Hall thruster.

With this in mind, there has been a focus in the past decade in developing more sophisticated, single-equation models for the turbulence. These have been inspired by the discovery that current-driven electrostatic turbulence may be the dominant driver for the transport in these systems.[18,19] Single equation closure efforts have focused on developing a differential equation based on quasilinear wave theory for how the energy in this turbulence evolves in the thruster.[8,11,17] Through a quasilinear approximation (c.f. Ref. 20), this energy then can be related to an effective anomalous collision frequency. While these efforts have served to improve in principle the fidelity of fluid-based simulations to the real system, they all have encountered a common paradox. In order to model the growth of the energy in the turbulence, it is necessary to invoke arguments from quasilinear theory about the rate at which the turbulence extracts energy with the background plasma. In particular, this rate of growth is assumed to be proportional to the dominant source of electron energy in the plasma, the Hall effect drift. As a result, single equation closure models that employ a classical quasilinear growth rate predict that the turbulence energy, its growth, and as a result, the effective collision frequency will all be highest where the Hall effect drift is the largest in the discharge. This location is coincident with the location of peak electric field. However, a strong anomalous collision frequency will promote cross-field electron transport, thereby leading to lower electric fields. Indeed, the electric field is highest in the thruster precisely where the anomalous collision frequency is lowest. As a result, unmodified single equation turbulence models predict a maximum anomalous collision frequency where in fact it should be the lowest in the discharge.

Faced with this limitation, it has been recognized that a single-equation model founded in simple quasilinear theory s not sufficient to accurately represent the effective collision frequency. Instead, it is necessary to develop a more nuanced method for accounting for how the rate at which the turbulence extracts energy from the plasma may be modified from its quasilinear form. To this end, i is commonly assumed that the wave growth is capped by a saturation process (e.g. ion wave trapping).[10,15,17] Similarly, ad hoc corrections have been applied to the rate of energy exchange of the turbulence with the background plasma. These correct for the fact that it is anticipated that nonlinear processes such as wave-wave coupling[21] or the formation of non-Maxwellian distributions[10] will modify the rate at which the plasma can extract energy from the plasma from its its quasilinear form. These corrections could be classified as 1.5 equation closures as they combine a full differential equation describing the wave energy density with an algebraic one for the rate at which the turbulence exchanges energy with the electrons. These 1.5 closure models have been able to demonstrate, with a small degree of calibration against data, a marked ability to match experimental measurements.

The success of these 1.5 models suggests that higher fidelity can be achieved with a closure model that does take into account both the evolution of the turbulence and the evolution of the rate at which the turbulence extracts energy from the plasma. A similar conclusion was reached a number of decades ago in the development of turbulence closure models for classical CFD problems.[22] This had led to the celebrated and widely used $k-\epsilon$ and $k-\omega$ models. The success of these models in capturing a wide range of physical processes in the CFD field recommends a similar two-equation problem for the Hall thruster closure problem. The goal of this work is derive such a model–one in which full differential governing equations are derived for both the total turbulent energy and the rate at which this turbulence extracts energy from the background plasma. To this end, this paper is organized in the following way. In the first section, we overview the assumptions in our approach and introduce the master equation we use for describing the transport of the plasma turbulence in our system, the plasma wave kinetic equation. We then take moments of this equation and combine them phenomenological arguments about non-linear processes include wave-wave coupling and kinetic effects to arrive at a two equation model. In the next section, we compare these models to experimental data. Finally, in the last section, we discuss the limitations of the model and how it may be extended to developing closed fluid models for the Hall thruster.

II. Two equation closure

The goal of this section is to determine for a Hall thrusters expressions both for the evolution of the total plasma turbulence energy density, W_T, as well as the average rate at which this turbulence extracts energy from the background plasma, $\langle\omega_i\rangle$. To this end, we make the following assumptions about the plasma turbulence consistent with simulations and experiments performed to date:

- The constituent waves in the spectrum are electrostatic and primarily oriented the Hall direction with acoustic-like dispersion given approximately by $\omega_r = c_s k_\theta + \vec{k}\cdot\vec{u}_i \approx c_s k_\theta$ where$\vec{k}$ is the wave vector of a given mode, k_θ is the component of the wavenumber in the azimuthal direction, c_s is the ion sound speed, $\vec{u}_i$ denotes the ion drift velocity, and ω_r is the frequency of the k^{th} wave.[23]

- Although the waves are oriented in the Hall direction, the group velocity is dominated by the ion drift, $\vec{v}_g = \partial\omega_r/\partial\vec{k} = \vec{u}_i$. This group velocity is independent of wave vector.

- The growth of the waves is driven by a collisionless interaction (inverse electron Landau or cyclotron damping) with the electron current in the Hall direction such that for the k^{th} mode we can write $\omega_i = \beta\omega_r M_e = \beta k_\theta c_s M_e$ where M_e denotes the electron Mach number in the Hall direction and β is a term that represents a potential departure of the electron distribution from Maxwellian.[10,15,17,24]

- The wave spectrum is turbulent characterized by propagating waves that are broadband and random phase.[19,25,26] Random phase averaging thus can be applied to determine average properties of the turbulence spectrum such as W_T and $\langle\omega_i\rangle$.

With these assumptions in mind, we introduce a master equation, the plasma wave-kinetic equation, to describe the evolution of wave properties in the plasma:[27]

$$\frac{\partial N_{\vec{k}}}{\partial t} + \vec{\nabla}\cdot\left(N_{\vec{k}}\vec{v}_g\right) - \nabla_{\vec{k}}\cdot\left(N_{\vec{k}}\ \nabla\omega_r\right) = N_{\vec{k}}\left(2\omega_i - 2\omega_{loss} + \mathcal{C}\left[N_{\vec{k}}, N_{\vec{k'}}\right]\right), \tag{1}$$

where $N_{\vec{k}}$ denotes the wave action density, ω_i is the growth rate or rate at which the k^{th} mode extracts energy from the plasma, ω_{loss} is the dissipation rate of the k^{th} wave, $\mathcal{C}\left[N_{\vec{k}}, N_{\vec{k'}}\right]$ denotes nonlinear processes including wave-wave interactions and wave-wave-particle interactions with the broadband plasma turbulence spectrum. Physically, Eq. 1 is the analog to the Boltzmann equation for plasma species where instead of a density distribution, $f_s(\vec{v}, \vec{x})$, of a given particle species as a function of velocity and position, we instead follow the density of plasmons (wavepackets), $N_{\vec{k}}(\vec{k}, \vec{x})$ as a function of wavevector and position. The left hand side thus represents the convection of the plasmons in real and wavevector space. The right hand side includes source terms for growth, dissipation, and effective "collision" term that represents the re-distribution of energy (the analog of thermalization of a plasma species) in the wave power spectrum through wave-wave interactions. In direct analogy to an interspecies collision operator, it is assumed that $\mathcal{C}\left[N_{\vec{k}}, N_{\vec{k}'}\right]$ conserves total plasmon number density[28,29] and drives the power spectrum of waves to an equilibrium shape as a function of $\vec{k}$.

Keeping with the interpretation of Eq. 1 as describing the evolution of the distribution function for the plasmons, we can arrive at expressions for the two quantities of interest by taking the zeroth and first moments with respect to the wavevector, $\vec{k}$. The zeroth moment yields

$$\frac{\partial N_T}{\partial t} + \vec{u}_i \cdot \nabla N_T = 2N_T \left(\langle \omega_i \rangle - \langle \omega_{loss} \rangle - \nabla \cdot \vec{u}_i\right), \tag{2}$$

where we have denoted averaged quantities $\langle x \rangle = \left[\int x N_{\vec{k}} d\vec{k}\right]/N_T$ and defined the total wave action over the turbulent as $N_T = \left[\int N_{\vec{k}} d\vec{k}\right]$. In order to relate Eq. 2 to the total wave energy density, W_T, in the power spectrum, we note that per the definition of wave action, $W_T = \left[\int \omega_r N_{\vec{k}} d\vec{k}\right]$. We thus can write

$$\frac{\partial W_T}{\partial t} + \vec{u}_i \cdot \nabla W_T = 2W_T \left(\langle \omega_i \rangle - \langle \omega_{loss} \rangle - \nabla \cdot \vec{u}_i - \frac{D}{Dt}\left[\ln\left[\langle \omega_r \rangle\right]\right]\right) \tag{3}$$

In order to simplify this expression further, we make the assumption that the rate in change of the logarithm of the average frequency of the spectrum is a weakly varying term compared to the growth and damping terms such that

$$\frac{\partial W_T}{\partial t} + \vec{u}_i \cdot \nabla W_T = 2W_T \left(\langle \omega_i \rangle - \langle \omega_{loss} \rangle - \nabla \cdot \vec{u}_i\right). \tag{4}$$

This result represents the evolution of the total wave energy density of the plasma turbulence in the volume with the left hand side representing the convection and the right hand side representing sources and sinks. It is the first equation in our two equation turbulence model.

The second equation for model describes the evolution of the major unknown in Eq. 4, $\langle \omega_i \rangle$, the average rate at which the waves are able to extract energy from the ambient plasma. To find a governing equation for this expression, we note that we are also looking for an evolution term of the form:

$$\begin{aligned} \frac{D}{Dt}\left[\langle \omega_i \rangle\right] &= c_s \beta M_e \frac{D}{Dt}\left[\langle k_\theta \rangle\right] + c_s \langle k_\theta \rangle \frac{D}{Dt}\left[\beta M_e\right] + \langle \omega_i \rangle \frac{D}{Dt}\left[\ln c_s\right] \\ \frac{D}{Dt} &= \frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla \end{aligned} \tag{5}$$

We can evaluate the first term on the right hand side by taking the first moment of Eq. 1 with respect to the wave vector and isolating the k_θ component. This yields approximately (Appendix):

$$\frac{D}{Dt}\left[\langle k_\theta \rangle\right] = 2\langle \omega_i \rangle \left(k_{max} - \langle k_\theta \rangle\right) - c_2 \langle \omega_r \rangle \langle k_\theta \rangle \frac{W_T}{n_e T_e}, \tag{6}$$

where n_e denotes the local electron density, T_e is the electron temperature, and we have introduced a constant coefficient, c_2. Eq. 6 represents the evolution of the average wavenumber of the spectrum. The first term on the right hand side physically reflects the fact that the linear growth of the turbulent spectrum is non-monotonic as a function of wavenumber and exhibits a maximum value. This maximum growth for the plasma turbulent in the Hall thruster is typically assumed to scale either the inverse of the Debye length or a wavenumber dictated by a cyclotron resonance.[10,30] Since new plasmons in the spectrum will be preferentially created at this wavelength corresponding to growth rate, the function of this first term is to drive the average value of the wavenumber in the spectrum to this value. The second term on the right hand side represents nonlinear, wave-wave coupling effects. These effects transport energy across the length scale of the spectrum, pushing it from shorter wavelengths to longer in an inverse energy cascade. This effect has been noted experimentally and numerically in Hall thrusters.[21,31]

Returning to Eq. 5, we recognize that since the waves of interest are driven by a collisionless interaction, we can invoke the quasilinear theory of electrostatic waves to write for ion acoustic like modes::[27]

$$\omega_i = \pi k c_s \left(\frac{T_e}{m_i}\right) f_e'\left(c_s\right), \tag{7}$$

where $f_e'\left(v_\theta\right)$ denotes the derivative of the electron velocity distribution function with respect to the electron velocity component in the Hall direction. Combining this result with our definition of ω_i, we thus recognize that

$$\beta M_e = \pi \left(\frac{T_e}{m_i}\right) f_e'\left(c_s\right). \tag{8}$$

In the limit that $f_e\left(v_\theta\right)$ is a Maxwellian distribution, we recover the classical result for current driven acoustic modes that $\beta = (m_e/m_i)\sqrt{\pi/2}$. However, we note that the presence of strong waves can lead to a deformation of the electron distribution function from Maxwellian.[10,24] While it is not possible to capture this deformation self-consistently without a full kinetic solution of the distribution function, we can make an approximation by employing scaling derived from quasilinear theory. Specifically, we consider the convective derivative:

$$\frac{D}{Dt}\left[\beta M_e\right] = \frac{\pi}{m_i}\left(\frac{D}{Dt}\left[T_e\right] f_e'\left(c_s\right) + T_e \frac{D}{Dt}\left[f_e'\left(c_s\right)\right]\right). \tag{9}$$

For the time derivative of the second term, we recognize that the rate of deformation of the velocity distribution function at the resonant condition, i.e. where the electron velocity is equal to the wave speed, $v_\theta = c_s$ is proportional to the wave energy density and the growth rate of the wave. This allows us to write by inspection (and leveraging Chapt. 15 of Ref. 27):

$$\frac{D}{Dt}\left[\beta M_e\right] = \left(\frac{\pi}{m_i}\frac{D}{Dt}\left[T_e\right] f_e'\left(c_s\right) - \pi \langle k_\theta \rangle c_s \frac{T_e}{m_i}\frac{W_T}{n_e T_e} f_e'\left(c_s\right)\right) \tag{10}$$

$$\frac{D}{Dt}\left[\beta M_e\right] = \beta M_e \left(\frac{D}{Dt}\left[\ln[T_e]\right] - \langle k_\theta \rangle c_s \frac{W_T}{n_e T_e}\right) \tag{11}$$

We again neglect the change in the logarithmic term compared to the second term on the right hand side to find the simplified result:

$$\frac{D}{Dt}\left[\beta M_e\right] = -\langle \omega_i \rangle \frac{W_T}{n_e T_e}. \tag{12}$$

The physical significance of this equation can be understood in the context of βM_e which we have defined to represent the change of the wave growth as the electron distribution is modified from a Maxwellian. As the rate at which the turbulence extracts energy from the plasma increases and as the actual level of plasma turbulence increases, we anticipate that the deviation from Maxwellian will increase. These effects should drive this parameter, $\beta M_e \rightarrow 0$ which is in fact satisfied by Eq. 12.

Armed with this result, we substitute Eq. 12 and Eq. 6 into Eq. 5 to find

$$\frac{D}{Dt}\left[\langle \omega_i \rangle\right] = 2\langle \omega_i \rangle \left(\omega_{max} - \langle \omega_i \rangle\right) - c_2 \frac{1}{M_e} \langle \omega_i \rangle^2 \frac{W_T}{n_e T_e}, \tag{13}$$

where we have folded all constants leading the term $\langle \omega_r \rangle \langle \omega_i \rangle \frac{W_T}{n_e T_e}$ into $c_2(\beta)$, which is now also a function of β. We make the approximation for simplicity, however, that this is a weak dependence, allowing c_2 to be constant. Physically, Eq. 13 shows how the average rate at which the turbulence extracts energy from the electrons has two counteracting influences. The first term on the right hand side drives the average growth of the spectrum to the frequency that corresponds to the maximum linear growth of the spectrum. The second term, on the other hand, represents nonlinear effects that include both the non-thermalization of the electron distribution function as well as an inverse energy cascade across length scales to lower frequency. Both effects work to lower the rate at which the turbulence can extract energy from the background plasma. As a final simplification, we make the assumption that the maximum linear growth rate will scale linearly with the ion plasma frequency.[10,15] This thus allows us to write

$$\frac{\partial W_T}{\partial t} + \vec{u}_i \cdot \nabla W_T = 2W_T \left(\langle \omega_i \rangle - \langle \omega_{loss} \rangle - \nabla \cdot \vec{u}_i\right). \tag{14}$$

$$\frac{\partial \langle \omega_i \rangle}{\partial t} + \vec{u}_i \cdot \nabla \langle \omega_i \rangle = 2\langle \omega_i \rangle \left(c_1 M_e \omega_{pi} - \langle \omega_i \rangle\right) - c_2 \frac{1}{M_e} \langle \omega_i \rangle^2 \frac{W_T}{n_e T_e}. \tag{15}$$

Taken together, these expressions represent a two-fluid model for the evolution of the total energy in the plasma turbulence and the evolution of the average rate at which this turbulence extracts energy from the background electrons.

Finally, we invoke quasilinear theory to relate these two average turbulence properties to a scalar that captures their impact of enhanced cross-field current, an anomalous collision frequency:

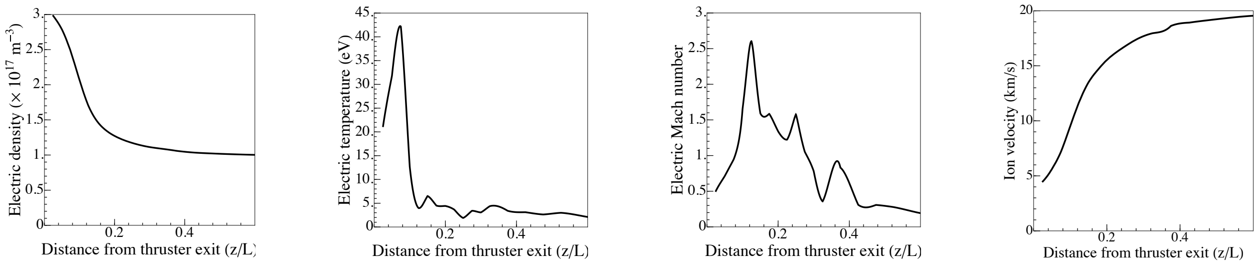
$$\nu_{AN} = c_3 \sqrt{\frac{m_i}{m_e}} \frac{1}{M_e} \langle \omega_i \rangle \frac{W_T}{n_e T_e}, \tag{16}$$

where we have introduced a third and final scaling constant, c_3. Eqs. 15 and 16 provide a set of governing relations with three fit parameters, c_1, c_2 and c_3 for describing the evolution of key properties of the plasma turbulence in the thruster plume and relating these properties to an effective collision frequency. We note that because of the need to introduce scaling parameters, this model is not self-consistent. The coefficients must be calibrated against data, as has been the case with all closure models applied to Hall thrusters to date. However, the goal, as also envisioned for two-fluid models for the CFD field, is that by introducing additional physical fidelity with this second equation, these coefficients may remain invariant or weakly invariant across multiple Hall thruster configurations and operating conditions. With this results, in mind, we turn in the next section to a proof of concept demonstration of the ability of this model, once calibrated, to re-create the average properties of the plasma turbulence as well as the anomalous electron collision frequency in an actual Hall thrusters.

III. Application of model to data

In order to calibrate the model we derived in the previous section, we consider data taken from a laboratory Hall thruster, the H9. The H9 is a 9-kW device that was designed by the NASA Jet Propulsion Laboratory and built in collaboration with the University of Michigan and the Air Force Research Laboratory.[32,33] The data we consider here and the methods used to generate this data are described in detail in Ref. 34. The key measurements we employ for this work were taken along the channel centerline at the 300 V and 15 A operating condition and include electron temperature, density, ion-neutral collision frequency, electron Mach number in the Hall direction, the axial ion velocity, the anomalous collision frequency, and the total wave energy density. We show the first four of these parameters as a function of distance from the thruster exit in Fig. 1. The measurements are referenced with respect to the exit plane of the thrust discharge chamber and are normalized by the discharge chamber length. As is consistent with typical Hall thruster operation, the plasma density decreases monotonically with position, and the electron temperature and Mach number both exhibit peaks. These are coincident with the region of peak magnetic and electric field. The ion velocity increases monotonically and then plateau. This is a consequence of the electrostatic acceleration of this species.

To evaluate Eqs. 15 and 16 for the plasma parameters shown in Fig. 1, we make the steady-state approximation such that we can neglect the time dependent terms, and we assume that ion-neutral collision are the dominant dissipation mechanism for the wave, i.e. $\langle \omega_{loss} \rangle = \nu_{in}$. We then integrate the equations numerically from the upstream boundary of the data shown in Fig. 1. We make the assumption that the plasma turbulence originates and begins to grow in this region. This is consistent with the physical interpretation that the turbulence is convected by the ion velocity. For the boundary conditions then, we select an initial wave energy density that is a small fraction of the thermal energy $W_T = 10^{-4} n_e T_e$. This reflects the fact that the turbulence originates from random fluctuations in the thermal background. For the initial average growth rate, we specify $\langle \omega_i \rangle = c_1 M_e \omega_{pi}$, This reflects the assumption that when the waves originate,

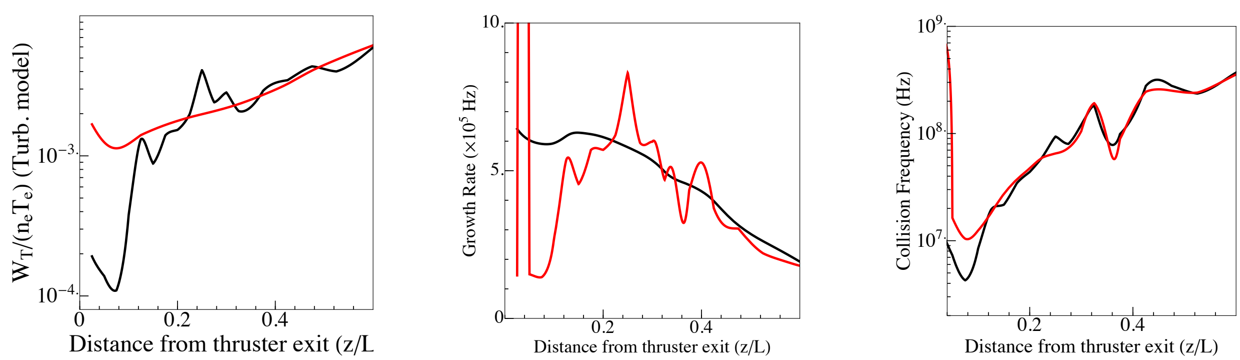


(a) (b) (c) (d)

Figure 1: Measurements from Ref.31 along centerline of the a) electron density, b) electron temperature, c) Mach number, and d) ion speed for the H9 Hall thruster operating at 300 V and 15 A. All distances have been normalized by L the length of the discharge chamber.

the average growth of the spectrum will occur at the maximum linear growth rate. With this prescriptions in mind, we show in Fig. 2 the relative wave energy density, rate of energy extraction from the plasma, and anomalous collision frequency. We also show on these plots the measured values inferred from Ref. **?**. While we did not measure the growth rate directly in this other work, we infer it here by solving Eq. **??** for $\langle\omega_i\rangle$ using the measured values of W_T and ν_{AN}. For the simulation, the calibration coefficients in this were $c_1 = 0.01$, $c_2 = 400$, and $c_3 = 130$ and were chosen to yield the best fit the measured collision frequency profile.

It is immediately evident from this result that the two-fluid model can be tuned to match with high fidelity the measured collision frequency. Similarly, the trends in the growth rate and the wave energy match over the majority of the length of the discharge (z/L = 0.1- 0.6). The prediction of the inflection point in at $z = 0.08L$ in the average growth followed by the gradual decrease in distance is particularly notable as it suggests this modified closure model captures a physical trend, the modification of the growth with position due to nonlinear effects. The gradual increase in the wave energy density downstream in both the two-closure and experimental result similarly is consistent with a saturated spectrum that has reached a value limited at a fraction of the thermal energy. With that said, there are marked deviations between the measured and simulated average growth and wave energy density. Most notably, the wave energy density and average growth drops moving upstream while the average growth rate's inflection is not nearly as pronounced as the experimental value. While these values are not reflected experimentally, we note that these two effects do combine in the two equation model to capture the inflection point exhibited in the anomalous collision frequency profile. This minimum in anomalous collision frequency is a common and critical element of Hall thruster discharged.



(a) (b) (c)

Figure 2: Plots along thruster channel centerline for (a) relative wave energy density, (b) average rate at which energy is extracted from plasma, and (c) the anomalous collision frequency. Red denotes the measurement from Ref. 31 while black is the result from the two fluid model. Calibration coefficients are. $c_1 = 0.01$, $c_2 = 400$, and $c_3 = 130$.

IV. Discussion

We discuss here the limitations of the two-equation model, its physical significance, and its traceability to classical fluid closures. First, we comment on the disagreement between the two equation model and the experimental results for the average turbulence properties n the upstream region ($z < 0.2L$). While the trends are qualitatively the same–exhibiting inflection points in both cases—the changes in energy and growth rate are more pronounced in the two-equation result. Notably, however, it is the product of these two values, *Eq.* **??**, that yields the anomalous collision frequency. So, even though the two-fluid equation model does not predict the same degree of change for each parameter, their products do seem to yield the same quantitative trends in collision frequency. From a practical perspective, the two-fluid closure still yields an accurate prediction. With that said, one possible explanation for this discrepancy may be traced to the choice of boundary conditions in solving the two equation model. We chose parameters that reflected the physical intuition that the plasma turbulence originates near $z = 0$, i.e. where we were able to collect data. In practice, in reviewing the experimental results, it appears as if the turbulence may already have reached a saturated state. Our two-fluid model predicts a rapid increase to reach saturation. This may also occur in the real system but potentially upstream of where we can measure.

Leaving aside the comparison with experiment, we note that we have only presented here model predictions for one set of calibration parameters. In practice, we found that this combination is not unique in yielding profiles qualitatively similar with those shown in Fig. 2. This opens the question of not only uniqueness but extensibility. Performing calibrations on additional datasets in principle would help narrow down the range of coefficients. However, it is not clear if the same coefficient combination will extend to different thrusters. Indeed, this is a known problem with all data-driven closure for Hall thrusters.[16] One of our goals in deriving a more physics-based model was to help introduce sufficient fidelity that these coefficients may be approximately constant. This has proved to be the case for many classical fluid turbulent models.[22] This question of this constancy, however, only will be resolved with additional data runs.

As a final comment, we remark on the parallels of the result we have derived here with similar closures commonly employed in CFD simulations. In structure and form, our result most closely resembles the $k - \epsilon$ turbulence model in which the turbulent energy, k, and the dissipation rate, ϵ, of the turbulence are modeled. In the case of these classical fluid, the energy cascades are to shorter length scales where the turbulent energy is ultimately dissipated. The dissipation mechanisms are poorly understand whereas the energy source for the turbulence, shear flow, are better known. As a result, the governing equation is formulated in terms of the unknown dissipation rate. In our case, we also model the turbulent energy; however, while we have a firm understanding of the dissipation mechanisms (collisions and ion-Landau damping), the rate at which energy is extracted from the plasma flow is not as well understood. This is because of the presence of non-linear effects that can give rise to inverse energy cascades and non-Maxwellian distributions. As a consequence, we have elected instead to develop a model for the growth rate of the turbulence, $\langle \omega_i \rangle$ instead of its dissipation.

Although the physical parameters that are modeled thus are different, in comparing classical fluid closures to the results we have derived here (Eq. 15), we see that that the equations are closely paralleled. Just as with CFD models (Chapter 4 of Ref. 22), our governing equation for the rate of change of energy in the turbulence has source terms that depend linearly and quadratically on the rate of change. We also have a term that approximates nonlinear effects related to the plasma energy diffusion across length scales just. A term diffusion similar in function appears in the $k - \epsilon$ closure that captures diffusion of the turbulence. With these parallels in mind, we should note that the same cautionary notes as found in classical reduced fidelity turbulence modeling also apply to our configuration. In particular, we have made a number of simplifications and approximations ("drastic surgery" in the language of Ref. 22) to the governing wave energy equation to arrive at a simplified fluid-like two-equation model. While our intention has been to ground this result in first-principles work to the highest degree possible, the resulting expressions are not entirely physics-based. We have sacrificed in some cases fidelity in favor of expediency and a tractable form that approximates the physical processes. The efficacy of making this trade, particular as to whether or not it permits extensibility to other systems, remains to be resolved. Indeed, all of the work we have done here as focused exclusively on the derivation and validation of the two equation model. This has been independent of a full fluid simulation. The ultimate test of the model efficacy will be to integrate the two equations into an existing Hall thruster numerical code to calibrate and benchmark against experimental measurements.

V. Conclusion

The goal of this work has been to introduce a two-fluid closure model, similar in form to the classical $k-\epsilon$ fluid turbulence model, that is capable of capturing the more nuanced effects associated with the development of plasma turbulence in a Hall thruster. In particular, by introducing a new equation for the evolution of the rate at which the turbulence extracts energy from the Hall current, we have been able to develop approximations for nonlinear effects including wave-wave coupling and the deviation of the electron distribution function from a thermalized state. We have applied this two-equation model to experimental data from a 9-kW class Hall thruster and shown that we can re-create salient trends in the plasma turbulence properties as well as their impact on the effective collision frequency of the electrons. Most notably, our results have shown that downstream of the region of peak electron Mach number, the turbulent energy growth is approximately saturated and the average growth rate decreases. We similarly have been able to re-create both qualitatively and quantitatively the measured electron collision frequency profile.

Despite the qualitative agreement between the model and experiment, we have also found limitations in its fidelity and use. In particular, we have noted discrepancies between the predicted turbulence properties and measurements in the vicinity of the peak Mach number. Interestingly, however, the deviations in wave energy an growth rate effectively cancel one another, thus still resulting in a calculated collision frequency that matched experiments with high fidelity. As another major limitation, we also have noted that the derived model still requires calibration against experimental data. This need to calibrate is a direct result of the fact that the model is inherently reduced fidelity. We are representing inherently kinetic effects in a fluid framework. While there are only three calibration coefficients, they are not unique, and it remains to be seen how extensible these will be to different thruster configurations.

Ultimately, the challenge of closure for Hall thrusters, remains a pressing issue for the practical, fluid-based modeling of these devices. The work we have presented here represents the next step in developing improved closure models. If properly and rigorously calibrated, the ultimate intention is that this tool can be used to build predictive capabilities.

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Appendix

We derive in the following a governing equation for the average component of the wave vector in the Hall direction. To this end, we we begin by taking the first moment with respect to $\vec{k}$ of the plasma wave kinetic equation (Eq. 1):

$$\frac{\partial\left[\langle\vec{k}\rangle N_T\right]}{\partial t}+\vec{\nabla}\cdot\left(N_T\langle\vec{k}\rangle\vec{v}_g\right)+N_T\langle\nabla\omega_r\rangle=2N_T\langle\omega_i\vec{k}\rangle-2\langle\vec{k}\rangle\langle\omega_{loss}\rangle N_T+N_T\langle\vec{k}\mathcal{C}\left[N_{\vec{k}},N_{\vec{k}'}\right]\rangle, \tag{17}$$

where we have assumed the loss rate is independent of wave vector, $\vec{k}$. We express this result in the non-conservative form by combining with the zeroth-order equation (Eq. 2):

$$N_T\frac{\partial\langle\vec{k}\rangle}{\partial t}+\vec{v}_g\vec{\nabla}\cdot\left(N_T\langle\vec{k}\rangle\right)+N_T\langle\nabla\omega_r\rangle=2N_T\left(\langle\omega_i\vec{k}\rangle-\langle\vec{k}\rangle\langle\omega_i\rangle\right)+N_T\langle\vec{k}\mathcal{C}\left[N_{\vec{k}},N_{\vec{k}'}\right]\rangle, \tag{18}$$

We are interested in examining the azimuthal component of this governing equation. To this end, we make the approximation that the second an third terms on the left hand side represent the spatial convection of the wavevector to write

$$\frac{\partial\langle k_\theta\rangle}{\partial t}+\vec{u}_i\cdot\nabla\langle k_\theta\rangle=2\left(\langle\omega_i k_\theta\rangle-\langle k_\theta\rangle\langle\omega_i\rangle\right)+\langle\vec{k}\mathcal{C}\left[N_{\vec{k}},N_{\vec{k}'}\right]\rangle_\theta. \tag{19}$$

This expression represents the evolution of the average wavenumber of the spectrum. Physically, it can be interpreted as the wavenumber most of the energy in the spectrum is located. In the absence of source terms (the right hand side is zero), the average wavenumber remains constant as the turbulence is convected through the thruster. The right hand side, however, behaves as sources and sinks which work to re-distribute the average value of this wavenumber. The expressions on the right hand side in the current form are not tractable. They require higher moments of the plasma wave kinetic equation (e.g. $\omega_i k_\theta$. Taking these higher moments naturally would introduce even higher moments, thus inviting a separate question of closure. Similarly, the form for the collisional operator, which works to re-distribute energy in the turbulent spectrum nonlinearly, remains unknown. To truncate the series, we make a number of simplifications inspired by physical arguments for these higher order terms. This is keeping with the spirit of more classical CFD closures where many of the sources terms are derived based on physical arguments rather than precise first principles.[22]

With this in mind, we first make the following equivalence:

$$(\langle\omega_i k_\theta\rangle - \langle k_\theta\rangle\langle\omega_i\rangle) = -2\langle\omega_i\rangle\left(\langle k_\theta\rangle - c_1 k_{max}\right) \tag{20}$$

where c_1 is a constant and k_{max} the wavenumber corresponding to the theoretical maximum growth for the electrostatic modes propagating in the plasma. Writing this expression in this can be interpreted as the wave equivalent of species drag that results from ionization. Physically, this form suggests that when a plasmon is created by drawing energy from the Hall effect drift, the plasmon is created at the wavelength corresponding to maximum growth. This is typically at much shorter length scales. This shifts the momentum in the power spectrum to this wavelength. Functionally, this term drives the average wavenumber to a value corresponding to the wavelength of maximum growth. For the second unknown component in Eq. 19, the collisional term, we draw inspiration from the first-principles collision operator that exists in the kinetic formulation of the Boltzmann equation for plasma species. Specifically, just as the interspecies collisional operator serves to thermalize the species, $\mathcal{C}$ includes the effects of nonlinear wave-wave and wave-wave-particle coupling that re-distribute the energy across length scales in the turbulence, driving the distribution to some equilibrium shape. With this parallel in mind, we note that the species collision operator is formulated as diffusion term in velocity space. Since $\vec{k}$ in the plasma wave kinetic is the analog of $\vec{v}$, we similarly represent $\mathcal{C}$ as a collision operator:

$$\langle\vec{k}\mathcal{C}\left[N_{\vec{k}}, N_{\vec{k}'}\right]\rangle_\theta = -\frac{\partial}{\partial k_\theta}\left[D_{k_\theta}\frac{\partial}{\partial k_\theta}N_{k_\theta}\right], \tag{21}$$

where we have introduced a diffusion operator for spread in wavenumber space. We propose a phenomenological form for this diffusion operator following the standard definition:

$$D_{k_\theta} = \frac{\Delta k_\theta^2}{\Delta t} \tag{22}$$

whereΔk_θ represents the movement across length-scales induced by nonlinear interactions and Δt is the characteristic time for each of these interactions. By dimensional analysis, we thus approximate

$$D_{k_\theta} = c_2 k_\theta^2\langle\omega_r\rangle\frac{W_T}{n_e T_e}, \tag{23}$$

where c_2 is a constant, and we have assumed the average movement in k-space scales with the value of k_θ and the characteristic time scale for interactions is the average frequency of the spectrum scaled by the wave energy density. These scalings reflect the fact that we anticipate smaller wavelength modes (higher k) will diffuse faster and that the rate of nonlinear wave interactions, intuitively, is moderated by the amplitude of the total energy in the turbulent spectrum. Armed with this result and Eq. 20, we thus can write Eq. 19 in the finalized form

$$\frac{\partial\langle k_\theta\rangle}{\partial t} + \vec{u}_i\cdot\nabla\langle k_\theta\rangle = 2\langle\omega_i\rangle\left(k_{max} - \langle k_\theta\rangle\right) - c_2\langle\omega_r\rangle\langle k_\theta\rangle\frac{W_T}{n_e T_e}. \tag{24}$$

References

[1]Boeuf, J.-P., "Tutorial: Physics and modeling of Hall thrusters," *Journal of Applied Physics*, Vol. 121, No. 1, 2019/09/14 2017, pp. 011101.

[2]Fife, J. M. and Martinez-Sanchez, M., "Two-Dimensional Hybrid Particle-in-Cell (PIC) Modeling of Hall Thrusters," *Proceedings of the 24th International Electric Propulsion Conference*, No. IEPC-95-24, 1995.

[3]Mikellides, I. G. and Katz, I., "Numerical Simulations of Hall-effect Plasma Accelerators on a Magnetic-Field-Aligned Mesh," *Physical Review E*, Vol. 86, No. 4, 2012, pp. 1–17.

[4]Barral, S., Makowski, K., Peradzyński, Z., Gascon, N., and Dudeck, M., "Wall material effects in stationary plasma thrusters. II. Near-wall and in-wall conductivity," *Physics of Plasmas*, Vol. 10, No. 10, 2003, pp. 4137–4152.

[5]Cappelli, M. A., Young, C. V., Cha, E., and Fernandez, E., "A zero-equation turbulence model for two-dimensional hybrid Hall thruster simulations," *Physics of Plasmas*, Vol. 22, No. 11, 2015.

[6]Garrigues, L., Hagelaar, G. J., Boniface, C., and Boeuf, J. P., "Anomalous conductivity and secondary electron emission in Hall effect thrusters," *Journal of Applied Physics*, Vol. 100, No. 12, 2006.

[7]Hofer, R., Katz, I., Mikellides, I., Goebel, D. M., Jameson, K. K., Sullivan, R. M., and Johnson, L. K., "Efficacy of Electron Mobility Models in Hybrid-PIC Hall Thruster Simulations," *Proceedings of the 44th AIAA Joint Propulsion Conference*, No. AIAA-2008-4924, 2008.

[8]Katz, I., Mikellides, I. G., Jorns, B. a., and Ortega, A. L., "Hall2De Simulations with an Anomalous Transport Model," *Proceedings of the 34th International Electric Propulsion Conference*, No. IEPC-2015-402, 2015.

[9]Koo, J. W. and Boyd, I. D., "Modeling of anomalous electron mobility in Hall thrusters," *Physics of Plasmas*, Vol. 13, No. 3, 2006, pp. 1–7.

[10]Lafleur, T., Baalrud, S. D., and Chabert, P., "Theory for the anomalous electron transport in Hall effect thrusters. I. Insights from particle-in-cell simulations," *Physics of Plasmas*, Vol. 23, No. 5, 2016.

[11]Mikellides, I., Ortega, A., Katz, I., and Jorns, B., "Hall2De simulations with a first-principles electron transport model based on the electron cyclotron drift instability," , No. AIAA-2016-4618.

[12]Morozov, A. I. and Savelyev, V. V., *Fundamentals of Stationary Plasma Thruster Theory*, Springer US, Boston, MA, 2000, pp. 203–391.

[13]Reza, M., Faraji, F., Andreussi, T., and Andrenucci, M., "A Model for Turbulence-Induced Electron Transport in Hall Thrusters," , No. IEPC-2017-367.

[14]Scharfe, M. K., Thomas, C. A., Scharfe, D. B., Gascon, N., Cappelli, M. A., and Fernandez, E., "Shear-Based Model for Electron Transport in 2D Hybrid Hall Thruster Simulations," *IEEE Transactions on Plasma Science*, Vol. 36, No. July, 2007, pp. 1–10.

[15]Katz, I., Lopez Ortega, A., Jorns, B., and Mikellides, I., "Growth and saturation of ion acoustic waves in hall thrusters," , No. AIAA-2016-4534.

[16]Jorns, B., "Predictive , Data-Driven Model for the Anomalous Electron Collision Frequency in a Hall Effect Thruster," *Plasma Sources Science and Technology*, Vol. 27, No. 104007, 2018.

[17]Lopez Ortega, A., Katz, I., and Chaplin, V., "A First-Principles Model Based on Saturation of the Electron Cyclotron Drift Instability for Electron Transport in Hydrodynamics Simulations of Hall Thruster Plasmas," *35th International Electric Propulsion ConferenceGeorgia Institute of Technology, Atlanta, Georgia * USA*, IEPC-2017-178.

[18]Adam, J. C., Héron, A., and Laval, G., "Study of stationary plasma thrusters using two-dimensional fully kinetic simulations," *Physics of Plasmas*, Vol. 11, No. 1, 2004, pp. 295–305.

[19]Tsikata, S., Lemoine, N., Pisarev, V., and Gŕsillon, D. M., "Dispersion relations of electron density fluctuations in a Hall thruster plasma, observed by collective light scattering," *Physics of Plasmas*, Vol. 16, No. 3, 2009, pp. 1–10.

[20]Davidson, R. C. and Krall, N. A., "Anomalous transport in high-temperature plasmas with applications to solenoidal fusion systems," *Nuclear Fusion*, Vol. 17, No. 6, 1977, pp. 1313–1372.

[21]Janhunen, S., Smolyakov, A., Chapurin, O., Sydorenko, D., Kaganovich, I., and Raitses, Y., "Nonlinear structures and anomalous transport in partially magnetized EB plasmas," *Physics of Plasmas*, Vol. 25, No. 1, 2019/09/14 2017, pp. 011608.

[22]Wilcox, D. C., *Turbulence Modeling for CFD*, DCW Industries, Incorporated, 1994.

[23]Cavalier, J., Lemoine, N., Bonhomme, G., Tsikata, S., Honoré, C., and Grésillon, D., "Hall thruster plasma fluctuations identified as the EB electron drift instability: Modeling and fitting on experimental data," *Physics of Plasmas*, Vol. 20, No. 8, 2013, pp. 082107–082107.

[24]Lafleur, T., Martorelli, R., Chabert, P., and Bourdon, A., "Anomalous electron transport in Hall-effect thrusters: Comparison between quasi-linear kinetic theory and particle-in-cell simulations," *Physics of Plasmas*, Vol. 25, No. 6, 2018.

[25]Brown, Z. A. and Jorns, B. A., "Dispersion relation measurements of plasma modes in the near-field plume of a 9-kW magnetically shielded thruster," , No. IEPC-2017-387.

[26]Brown, Z. and Jorns, B. A., "Spatial Evolution of Plasma Waves in the Near-field of a Magnetically Shielded Hall Thruster," *54th AIAA Joint Propulsion Conference2018 Joint Propulsion Conference*, 2018, pp. AIAA–2018–4423.

[27]Stix, T. H., *Waves in Plasmas*, American Inst. of Physics, 1992.

[28]Sagdeev, R. Z. and Galeev, A., *Nonlinear plasma theory*, Frontiers in physics, W. A. Benjamin, 1969.

[29]Tsytovich, V. N. and Haar, D. T., *An Introduction to the Theory of Plasma Turbulence: International Series of Monographs in Natural Philosophy*, Elsevier Science, 2016.

[30]Koshkarov, O., Smolyakov, A., Raitses, Y., and Kaganovich, I., "Self-Organization, Structures, and Anomalous Transport in Turbulent Partially Magnetized Plasmas with Crossed Electric and Magnetic Fields," *Physical Review Letters*, Vol. 122, No. 18, 05 2019, pp. 185001–.

[31]Brown, Z. and Jorns, B., "Spatial Evolution of Small Wavelength Fluctuations in a Hall Thruster," *Submitted.*, 2019.

[32]Hofer, R., Cusson, S., Lobbia, R., and Gallimore, A., "The H9 Magnetically Shielded Hall Thruster," *Proceedings of the 35th International Electric Propulsion Conference*, No. IEPC-2017-232, 2017.

[33]Cusson, S. E., Hofer, R., Lobbia, R., Jorns, B., and Gallimore, A., "Performance of the H9 Magnetically Shielded Hall Thrusters," *Proceedings of the 35th International Electric Propulsion Conference*, No. IEPC-2017-239, 2017.

[34]Brown, Z., Dale, E., and Jorns, B., "Experimental Correlation between Anomalous Electron Collision Frequency and Plasma Turbulence in a Hall E ect Thruster," *36th International Electric Propulsion Conference, University of Vienna, Austria, IEPC-2019-843 September 15-20, 2019*, IEPC-2019-843.